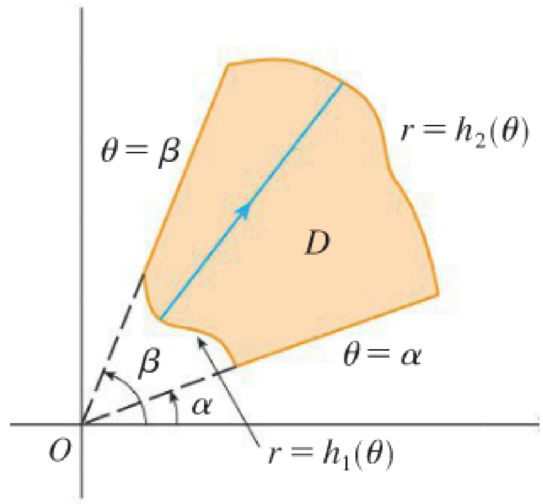
Sec 15.3 Double Integrals in Polar Coordinates



The planar region D is described in polar coordinates as follows: $\alpha \le \theta \le \beta$ and $h_1(\theta) \le r \le h_2(\theta)$.

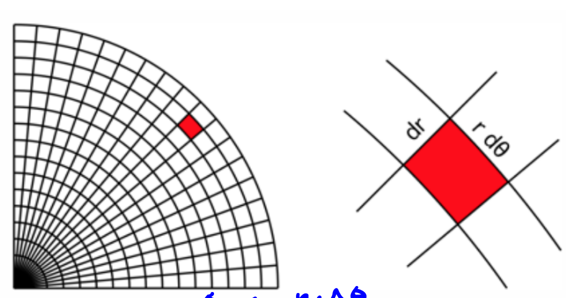
$x = r\cos(\theta)$
$y = r\sin(\theta)$
$x^2 + y^2 = r^2$

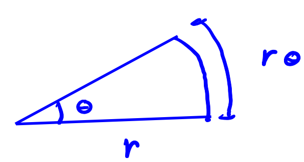
Theorem.

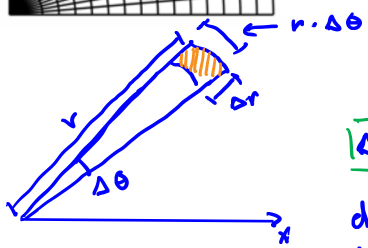
$$\overbrace{\iint_{\mathcal{D}} f(x,y)\ dA}^{\text{In cartesian or rectangular coords}} = \iint_{\mathcal{D}_{\theta,r}} f(r\cos\theta, r\sin\theta)\ r\,dr\,d\theta.$$

where $\mathcal{D}_{\theta,r}$ is the region $\mathcal{D}$ described in polar coordinates.

Intuitive Idea:





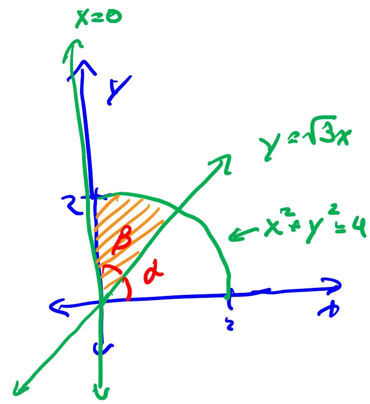


$\boxed{\Delta A = (\Delta r)(r\Delta\theta)}$

$dA = dr \cdot r d\theta$

$dA = r dr d\theta$

Ex1. Compute $\iint_{\mathcal{D}} 4 - x^2 - y^2 \, dA$, where $\mathcal{D}$ is the region in the I-quadrant bounded by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $y = \sqrt{3}x$.



Bounds for θ: $\alpha \le \theta \le \beta$

$\alpha = ?$ $\tan(\alpha) = \frac{y}{x} = \frac{\sqrt{3}x}{x} = \frac{\sqrt{3}}{1}$

$\Rightarrow \alpha = \pi/3$

$\beta = ?$ $\beta = \pi/2$

so, $\pi/3 \le \theta \le \pi/2$.

Bounds for r:

enters: $r = 0$

leaves: $x^2 + y^2 = 4$

$\Rightarrow r^2 = 4$

$\Rightarrow r = 2$

so, $0 \le r \le 2$

then $D = \{(r,\theta): 0 \le r \le 2, \ \pi/3 \le \theta \le \pi/2\}$.

Evaluate

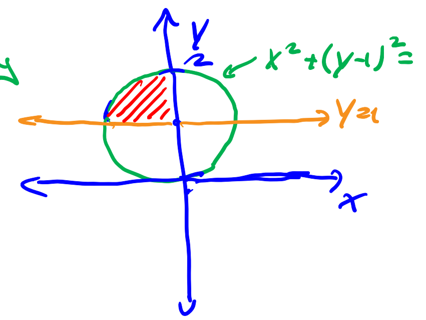
$$\iint_D 4 - x^2 - y^2 \, dA = \int_{\pi/3}^{\pi/2} \int_0^2 (4 - r^2) \overbrace{(r)}^{\text{from previous page}} \, dr \, d\theta$$

Inner: $\int_0^2 4r - r^3 \, dr = \left(2r^2 - \frac{r^4}{4}\right)\Big|_0^2 = (8-4) - (0-0) = 4$.

outer: $\int_{\pi/3}^{\pi/2} (4) \, d\theta = 4\theta \Big|_{\pi/3}^{\pi/2} = 4(\pi/2) - 4(\pi/3) = 4(\pi/6) = \frac{2\pi}{3}$

so, $\iint_D 4 - x^2 - y^2 \, dA = \frac{2\pi}{3}$

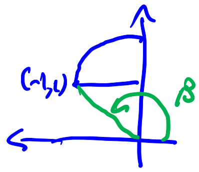
Ex2. Evaluate $\iint_{\mathcal{D}} xy\, dA$ where $\mathcal{D}$ is the region in the II-quadrant bounded above by the circle $x^2 + (y-1)^2 = 1$ and bounded below by the line $y = 1$.



Bounds for θ: $\alpha \le \theta \le \beta$

$\alpha = ?$ $\quad \pi/2$

$\beta = ?$ $\quad \frac{3\pi}{4}$



$\tan^{-1}\left(\frac{1}{-1}\right)$

so, $\frac{\pi}{2} \le \theta \le \frac{3\pi}{4}$

Bounds for r:

$$\begin{cases} \text{Enters: } y=1 \Rightarrow r\sin(\theta)=1 \Rightarrow \boxed{r = 1/\sin(\theta)} \\ \text{Leaves: } x^2+(y-1)^2=1 \Rightarrow x^2+y^2-2y+\cancel{1}=\cancel{1} \end{cases}$$

$$r^2 - 2r\sin(\theta) = 0$$

$$r(r-2\sin(\theta)) = 0$$

$$r=0 \quad \text{or} \quad r = 2\sin\theta$$

so, $\frac{1}{\sin\theta} \le r \le 2\sin\theta$

then, $D = \left\{(r,\theta) : \frac{1}{\sin\theta} \le r \le 2\sin\theta,\ \frac{\pi}{2} \le \theta \le \frac{3\pi}{4}\right\}$.

Evaluate:

remember r^4

$$\iint_D xy\,dA = \int_{\pi/2}^{\frac{3\pi}{2}} \int_{\frac{1}{\sin\theta}}^{2\sin\theta} (r\cos\theta)(r\sin\theta)(r)\,dr\,d\theta$$

Inner: $\displaystyle\int_{1/\sin\theta}^{2\sin\theta} r^3\cos\theta\sin\theta\,dr = \frac{r^4}{4}\cos\theta\sin\theta \Big|_{r=1/\sin\theta}^{r=2\sin\theta}$

$$= 4\sin^4\theta\cos\theta\sin\theta - \frac{1}{4\sin^4\theta}\cos\theta\sin\theta$$

$$= 4\sin^5\theta\cos\theta - \frac{\cos\theta}{4\sin^3\theta}$$

Outer: $\displaystyle\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} 4\sin^5\theta\cos\theta - \frac{\cos\theta}{4\sin^3\theta}\,d\theta = \left(4\,\frac{\sin^6\theta}{6} + \frac{1}{8\sin^2\theta}\right)\Big|_{\pi/2}^{3\pi/2}$

$u = \sin\theta$

$$= \ldots = \boxed{\frac{-11}{24}}$$

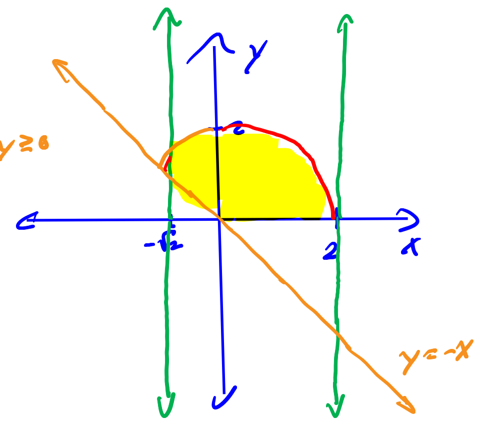
practice: check!

Ex3. Write the following sum

$$I = \int_{-\sqrt{2}}^{0}\int_{-x}^{\sqrt{4-x^2}} \exp(x^2+y^2)\;dy\,dx \;+\; \int_{0}^{2}\int_{0}^{\sqrt{4-x^2}} \exp(x^2+y^2)\;dy\,dx$$

as one double integral. Then evaluate the double integral.

$$\begin{cases} -\sqrt{2} \le x \le 0 \\ \underbrace{-x}_{y=-x} \le y \le \underbrace{\sqrt{4-x^2}}_{x^2+y^2=4,\ y\ge 0} \end{cases}$$



$$\begin{cases} 0 \le x \le 2 \\ 0 \le y \le \sqrt{4-x^2} \end{cases}$$

$x^2+y^2=4 \Rightarrow r^2=4$
$\Rightarrow r=2$

remember "r"

then $$I = \int_0^{\frac{3\pi}{4}}\int_0^{2} e^{r^2}(r)\,dr\,d\theta$$

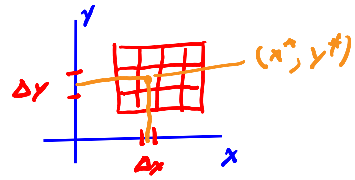
$$= \dots = \frac{3\pi(e^4-1)}{8}$$

Sec 15.4: Applications of Double Integrals

2D-Mass, Moments and Center of Mass

If $\delta(x, y)$ is the density(mass per unit area) of an object occupying a region $\mathcal{R}$ in the xy-plane, the integral of δ over $\mathcal{R}$ gives the **mass** of the object.

$$m = \iint\limits_{\mathcal{R}} \underbrace{\delta(x, y)}_{\text{density}}\, dA$$



First moments about the coordinate axes:

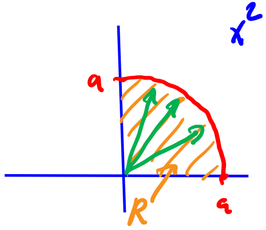
$$M_y = \iint\limits_{\mathcal{R}} x \cdot \delta(x, y)\, dA, \qquad M_x = \iint\limits_{\mathcal{R}} y \cdot \delta(x, y)\, dA$$

Center of mass:

$$\bar{x} = \frac{M_y}{m}, \qquad \bar{y} = \frac{M_x}{m}$$

Centroid: When the density of a plate is constant, the center of mass is called **centroid** of the object.

Ex1. Find the centroid of the region cut from the first quadrant by the circle $x^2 + y^2 = a^2$.



$x^2+y^2=a^2$
$r^2=a^2$
$r=a$

density: $\delta(x,y) = K$ (constant)

$$\text{mass} = \iint\limits_R K\, dA = K \iint\limits_R 1\, dA = K\frac{\pi (a)^2}{4}$$

$$M_y = \iint\limits_R x \cdot K \cdot dA = K\iint\limits_R y\, dA = K\int_0^{\pi/2}\int_0^a r\cos\theta\; r\, dr\, d\theta = \dots \text{ evaluate!}$$

$$M_y = K\, a^3/3$$

$$M_x = \iint\limits_R y \cdot K \cdot dA = K\iint\limits_R y\, dA = K\int_0^{\pi/2}\int_0^a r\sin\theta\; r\, dr\, d\theta = \dots \text{ evaluate!}$$

$$M_x = K\, a^3/3$$

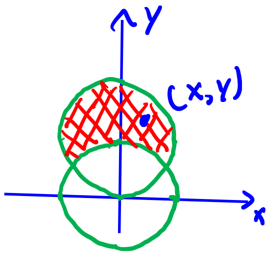
so, the centroid is:

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{\text{mass}}, \frac{M_x}{\text{mass}}\right) = \left(\frac{\cancel{K}\, a^3/3}{\cancel{K}\pi a^2/4}, \frac{\cancel{K}\, a^3/3}{\cancel{K}\pi a^2/4}\right) = \left(\frac{4a}{3\pi}, \frac{4a}{3\pi}\right).$$

$x^2+(y-1)^2=1$

Ex2. A lamina occupies the region inside the circle $x^2+y^2=2y$ but outside the circle $x^2+y^2=1$. Find the center of mass if the density at any point is inversely proportional to its distance from the origin.

density: $\delta(x,y)=\dfrac{k}{\sqrt{x^2+y^2}}$; $(\bar{x},\bar{y})=\left(0,\dfrac{M_x}{\text{mass}}\right)$



Bounds for r: $\begin{cases} \text{enters: } x^2+y^2=1 \Rightarrow \boxed{r=1} \\ \text{leaves: } x^2+y^2=2y \Rightarrow r^2=2r\sin\theta \end{cases}$

$\Rightarrow \boxed{r=2\sin\theta}$

Bounds for θ: Intersection between $r=1$ and $r=2\sin\theta$

$\Rightarrow 2\sin\theta=1 \Rightarrow \sin\theta=\frac{1}{2} \Rightarrow \theta=\frac{\pi}{6},\ \pi-\frac{\pi}{6}$

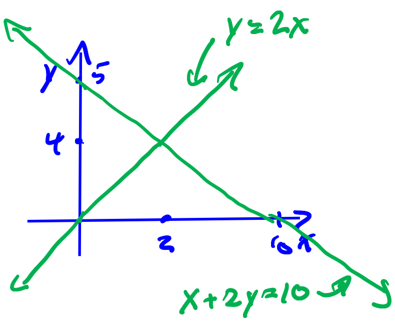
$$\text{mass}=\iint_R (\text{density})\,dA=\int_{\pi/6}^{5\pi/6}\int_1^{2\sin\theta}\frac{k}{\cancel{r}}\cancel{r}\,dr\,d\theta=\ldots=k\left[2\sqrt{3}-\frac{2\pi}{3}\right]$$

$$M_x=\iint_R y\,(\text{density})\,dA=\int_{\pi/6}^{5\pi/6}\int_1^{2\sin\theta} r\sin\theta\,\frac{k}{\cancel{r}}\cancel{r}\,dr\,d\theta=\ldots=k\sqrt{3}$$

then $(\bar{x},\bar{y})=\left(0,\dfrac{M_x}{\text{mass}}\right)=\left(0,\dfrac{k\sqrt{3}}{k\left[2\sqrt{3}-\frac{2\pi}{3}\right]}\right)=\left(0,\dfrac{\sqrt{3}}{\left(2\sqrt{3}-\frac{2\pi}{3}\right)}\right)$

TO DO

Find the mass and the center of mass of the lamina that occupies the region $\mathcal{R}$ enclosed by the lines $y=0$, $y=2x$, and $x+2y=10$; and has density function $\delta(x,y)=x$.



$$\text{mass}=\iint_R (\text{density})\,dA$$

$$=\int_0^4\int_{y/2}^{10-2y} x\,dx\,dy$$